Nonlinear theory of magnetic fluctuations in random flow: The Hall effect

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A nonlinear theory of magnetic fluctuations excited by random flow of a conducting fluid is developed. A mechanism of amplification of magnetic fluctuations in the presence of zero mean field, proposed by Zeldovich, is applied to the theory by means of a nonlinear equation derived from the induction equation; the nonlinearity is associated with the Hall effect. To derive the nonlinear equation we used a method [S. A. Molchanov, A. A. Ruzmaikin, and D. D. Sokoloff, Sov. Phys. Usp. **28**, 307 (1985)] the main idea of which is to replace the magnetic diffusion by the Wiener process. The diffusive motion is described by means of an average over an ensemble of random Wiener trajectories. The nonlinear equation describes the evolution of the correlation function of the magnetic field and resembles the Schrödinger equation except for a variable mass and the absence of the imaginary unit in the time-derivative term. The local spatial distribution of the magnetic field is intermittent: the field is concentrated inside flux tubes separated by regions with weak fields. In the limit of large Reynolds number the formulation is amenable to treatment by a modified WKB method. The general properties of the nonlinear stationary asymptotic solution are confirmed by the numerical solution. The results obtained are of interest for the ionosphere of Venus.

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I. INTRODUCTION

Investigations of an origin and evolution of magnetic fluctuations are important from the point of view of various cosmic and laboratory applications (see, e.g., [2–4]). Many recent studies have focused on deterministic (see, e.g., [5–8]) and random (see, e.g., [4,9–15] and references therein) flows which can excite the magnetic fluctuations.

A mechanism of generation of magnetic fluctuations with zero mean magnetic field was proposed by Zeldovich (see, e.g., [16,4]). In Fig. 1 we illustrate how the mechanism operates. An original loop of magnetic field is stretched [Fig. 1(b)], twisted [Fig. 1(c)] and then folded [Fig. 1(d)]. These nontrivial motions are three dimensional and result in an amplification of the magnetic field. Magnetic diffusion leads to reconnection of the field at an X point. If the turbulent flows of conductive fluids tend to be one dimensional, for example, because of the presence of an external field, magnetic fluctuations are not generated (see, e.g., [4]).

As follows from [11,17] the turbulent magnetic field is concentrated within flux ropes separated by regions of weak field. Statistical properties of the turbulent magnetic field $\mathbf{B}$ can be described by means of the correlation function $W(r,t) = \langle B(\mathbf{x},t)B(\mathbf{y},t)\rangle$. Here the angular brackets mean statistical averaging and B is the projection of magnetic field $\mathbf{B}$ on the direction $\mathbf{r} = \mathbf{x} - \mathbf{y}$. A linear equation describing the evolution of the correlation function W of the magnetic field was derived differently in [1,18–20] for a prescribed incompressible turbulent velocity field $\mathbf{u}$:

$$\langle \mathbf{u}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{y},t+\tau)\rangle = l_0 u_0 f(r)\delta(\tau) , \qquad (1)$$

where, for example, $f(r) = \exp(-r^2)$, $\delta(\tau)$ is the delta function, τ is the correlation time, l_0 is the main scale of turbulent hydrodynamic pulsations, and u_0 is the characteristic value of the turbulent velocity $\mathbf{u}$. The results

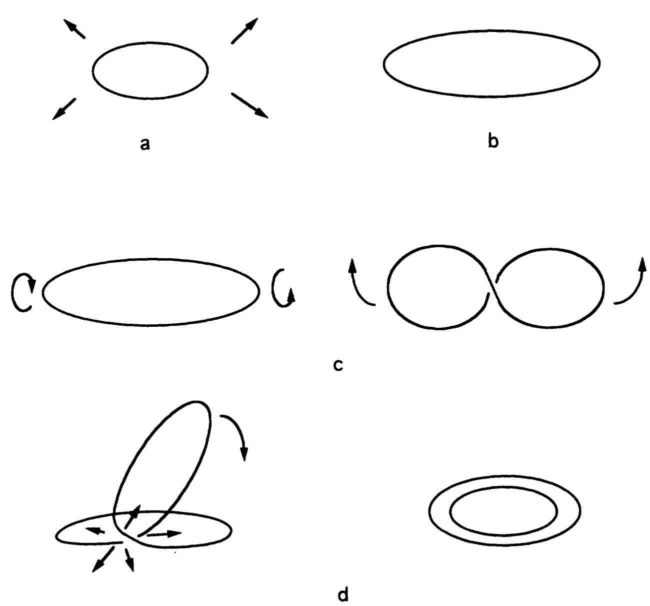

FIG. 1. The mechanism of amplification of magnetic fluctuations [16].

remain valid also for a velocity field with a finite correlation time if the statistical characteristics of magnetic field vary slowly in comparison with the correlation time [21].

However, a level of the magnetic fluctuations cannot be estimated from the linear model. In this paper a nonlinear theory of magnetic fluctuations excited by random flows of a conducting fluid is proposed. We take into account the nonlinearity in the induction equation caused by the Hall effect. An asymptotic stationary solution of the nonlinear equation for the correlation function of the magnetic field in the limit of large Reynolds number is obtained. It is confirmed by the numerical simulation. On the basis of this theory it is possible to explain magnetic field observations in the ionosphere of Venus [22,23].

II. THE HALL EFFECT

The nonlinear equation describing the evolution of the correlation function W for the magnetic field can be derived from the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v}_i \times \mathbf{B} + \frac{c}{4\pi e n} \mathbf{B} \times (\nabla \times \mathbf{B}) - \eta \nabla \times \mathbf{B} \right) , \tag{2}$$

where $\eta = c^2/4\pi\sigma$ is the magnetic diffusivity, σ is the electrical conductivity, $\mathbf{v}_i$ is the ion velocity, e is the electron charge, n is the electron number density, and c is the speed of light. The second term in Eq. (2) describes the Hall effect. The induction equation (2) is derived from the Maxwell equations and Ohm's law.

Let us discuss the Hall effect. We consider three-fluid magnetohydrodynamics for electrons, ions, and neutral particles. The momentum equations for electrons and ions are given by (see, e.g., [24,25])

$$m_i n_i \left(\frac{d\mathbf{v}_i}{dt} \right) = - \nabla p_i + zen_i \mathbf{E} + \frac{zen_i}{c} (\mathbf{v}_i \times \mathbf{B}) + \frac{m_e n}{\tau_{ei}} (\mathbf{v}_e - \mathbf{v}_i) + \frac{m_i n_i}{\tau_{in}} (\mathbf{u} - \mathbf{v}_i) , \tag{3}$$

$$m_e n \left(\frac{d\mathbf{v}_e}{dt} \right) = - \nabla p_e - en\mathbf{E} - \frac{en}{c} (\mathbf{v}_e \times \mathbf{B}) - \frac{m_e n}{\tau_{ei}} (\mathbf{v}_e - \mathbf{v}_i) + \frac{m_e n}{\tau_{en}} (\mathbf{u} - \mathbf{v}_e) , \tag{4}$$

where

$$\frac{d\mathbf{v}_i}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} ;$$

$\mathbf{v}_e, \mathbf{v}_i$, and $\mathbf{u}$ are the electron, ion, and neutral particle velocities, respectively; m_e and m_i are the electron and the ion mass, p_e and p_i are the electron and the ion pressure; τ_{in}, τ_{en}, and τ_{ei} are the ion-neutral, electron-neutral, and electron-ion collision times, respectively; ze is the ion charge, and n_i is the ion number density. A relationship $zn_i = n$ is due to the electrical quasineutrality of plasma and $\mathbf{E}$ is the electric field.

We neglect the inertia of electrons $m_e n(d\mathbf{v}_e/dt)$ in Eq. (4) because $m_e \ll m_i$. Ohm's law follows from Eq. (4):

$$\mathbf{j} = \sigma \left(\mathbf{E} + \frac{1}{4\pi e n} \mathbf{B} \times (\nabla \times \mathbf{B}) + \frac{1}{c} \mathbf{v}_i \times \mathbf{B} + \frac{\nabla p_e}{en} \right) , \tag{5}$$

where the electric current is $\mathbf{j} = en(\mathbf{v}_e - \mathbf{v}_i), \nabla \times \mathbf{B} = (4\pi/c)\mathbf{j}$, and we consider for simplicity the case $\tau_{ei} \ll \tau_{en}$. So the conductivity is $\sigma = e^2 n \tau_{ei}/m_e$. The second term in Ohm's law describes the Hall effect. The sum of Eqs. (3) and (4) yields

$$m_i n_i \left(\frac{d\mathbf{v}_i}{dt} \right) = -\nabla p + \frac{1}{c} (\mathbf{j} \times \mathbf{B}) + \frac{m_i n_i}{\tau_{in}} (\mathbf{u} - \mathbf{v}_i) , \tag{6}$$

where $p = p_i + p_e$ and we take into account that $m_e \ll m_i$.

The second term in Eq. (6) is quadratic in terms of the magnetic field and describes the influence of the magnetic field on the motion of plasma. It follows from Eqs. (2), (5), and (6) that this term corresponds to the cubic nonlinearity ($\sim B^3 \tau_i / 4\pi m_i n_i l_B$) in terms of the magnetic field in the induction equation. Here τ_i is the characteristic time of the ion component of the plasma and l_B is the characteristic scale of the magnetic field variations. On the other hand, the nonlinearity in the induction equation caused by the Hall effect [the second term in Eq. (2)] is a quadratic nonlinearity in terms of the magnetic field.

Now let us compare these two kinds of nonlinearity. First, we consider the case $\tau_{in} \gg \tau_i$. It follows from Eq. (6) that a variation of the ion velocity δv_i under the influence of the generated magnetic field is

$$\delta v_i \sim \frac{\tau_i}{m_i n_i c} |\mathbf{j} \times \mathbf{B}| .$$

The cubic nonlinearity is not as effective as the quadratic if

$$|\delta \mathbf{v}_i \times \mathbf{B}| \ll \left| \frac{c}{4\pi e n} \mathbf{B} \times (\nabla \times \mathbf{B}) \right| .$$

It yields the following criterion:

$$\omega_{Hi} \tau_i \ll 1 , \tag{7}$$

where $\omega_{Hi} = eB/(m_i c)$ is the ion gyrofrequency. Note that an external force can determine the characteristic time τ_i of the ion component of plasma.

Now we study the case when the ion time τ_i is much longer than τ_{in}. Therefore, the solution of Eq. (6) is given by

$$\mathbf{v}_i = \mathbf{u} - \frac{\tau_{in}}{m_i n_i} \left(\nabla p + \frac{1}{4\pi} \mathbf{B} \times (\nabla \times \mathbf{B}) \right) . \tag{8}$$

For an incompressible flow $\nabla \cdot \mathbf{v}_i = \nabla \cdot \mathbf{u} = 0$ and the pressure can be determined from Eq. (8):

$$\Delta p = -\frac{1}{4\pi} \nabla \cdot [\mathbf{B} \times (\nabla \times \mathbf{B})] .$$

It follows from (8) that the variation of the ion velocity δv_i under the influence of the generated magnetic field is

$$\delta v_i \sim \frac{\tau_{in}}{m_i n_i c} |\mathbf{j} \times \mathbf{B}| .$$

Therefore, in this case the cubic nonlinearity is not as effective as the quadratic if

$$\omega_{Hi}\tau_{in} \ll 1 . \qquad (9)$$

The effect of an external force $\mathbf{F}$ on plasma is reduced to a change of the ion velocity by a value $\tau_{in}|\mathbf{F}|/m_i n_i$. It results in the appearance of an additional external emf in the induction equation (2). It leads to the generation of the seed magnetic field. In the intermediate case, when $\tau_i \sim \tau_{in}$, the criteria (7) and (9) coincide.

In this paper attention is restricted to the quadratic nonlinearity; the cubic one will be considered in a separate paper. Therefore, in Ohm's law (5) and in the induction equation (2) we replace the ion velocity $\mathbf{v}_i$ by $\mathbf{u}$.

When an external regular (mean) magnetic field is superimposed on the plasma, it can magnetize electron component of the plasma if $\omega_{He}\tau_{ei} \gg 1$ (see, e.g., [24,25]). Here ω_{He} is the electron gyrofrequency. In this case the magnetic fluctuations cannot be generated because the motions of electrons tend to be one dimensional. The physical reasoning behind the impossibility of the generation lies in the need for three dimensionality of the flow of the plasma. Note that the magnetic Reynolds number defined on the ion component of the plasma can be lower than the threshold of the self-excitation of the magnetic fluctuations (the threshold value of the magnetic Reynolds number $R_m^{\rm cr} \approx 66$; see Sec. IV). Therefore, in spite of the ion component of the plasma being not magnetized yet by the external regular field, the generation of the magnetic fluctuations cannot occur for the low magnetic Reynolds number. This case seems to be typical for the ionosphere of Venus (see the discussion in Sec. VI).

In the next section we will derive the nonlinear equation for the correlation function for magnetic field from the induction equation after taking into account the Hall effect.

III. THE NONLINEAR EQUATION

To derive the equation for the correlation function $W(r,t)$ we used a method developed in [1,20]. The main idea of the method is to replace the magnetic diffusion by the Wiener process (see, e.g., [4]). The diffusive motion is described by means of an average over an ensemble of random Wiener trajectories. A Wiener random process is defined by the properties

$$M\{\mathbf{w}_t\} = 0 ,$$
$$M\{(\mathbf{w}_t)_i(\mathbf{w}_t)_j\} = t\delta_{ij} ,$$

where M is the mathematical expectation over the Wiener paths. It means that $w_t \sim t^{1/2}$ for $t \to 0$, so the Wiener process can describe diffusion. We use the Lagrangian solution of the induction equation (see, e.g., [1]). The problem reduces to the analysis of the field evolution $\mathbf{B}(t,\mathbf{r})$ along the Wiener path ξ_t:

$$\xi_t = \mathbf{x} - \int_0^t \mathbf{v}(t-s,\xi_s)\,ds + (2\eta)^{1/2}\mathbf{w}_t . \qquad (10)$$

Here $\mathbf{v} = \mathbf{u} - (c/4\pi en)\nabla \times B$. Equation (10) describes a set of the random trajectories which pass through the point $\mathbf{x}$ at time t. Because $\mathbf{w}_t$ is a Wiener process, the initial coordinates ξ_t are random. Without the magnetic diffusion ($\eta = 0$) the Wiener paths coincide with Lagrangian trajectories and ξ_t is not random. The approach is similar to the method of Feynman integrals over trajectories in quantum mechanics and does not require the assumption of the Gaussian statistics for the turbulence. This feature is most important for the nonlinear problem because the statistical distribution of the effective velocity $\mathbf{v} = \mathbf{u} - (c/4\pi en)\nabla \times \mathbf{B}$ [see Eq. (2)] cannot be given.

The solution of the induction equation (2) with the initial condition $\mathbf{B}(t=t_0,\mathbf{x}) = \mathbf{B}_0(\mathbf{x})$ is given by

$$B_i(t,\mathbf{x}) = M\{G_{ij}(t_0,t,\mathbf{x},\xi_t)B_{0j}(\xi_t)\} , \qquad (11)$$

where G_{ij} is the Green's function determined by the equation

$$\frac{d}{ds}G_{ij}(t_0,t-s,\mathbf{x},\xi_s) = -G_{kj}\frac{\partial v_i(t-s,\xi_s)}{\partial x_k} , \qquad (12)$$

with the initial condition for $t = s : G_{ij}(t_0,0,\mathbf{x},\mathbf{x}) = \delta_{ij}$ [1]. According to Eqs. (11) and (12), in a short time Δt the magnetic field varies as

$$B_i(t+\Delta t,\mathbf{x}) = M\left[\left(\delta_{ij} + \frac{\partial v_i(t,\mathbf{x})}{\partial x_j}\Delta t\right)B_j(\xi_t)\right] . \qquad (13)$$

Let us calculate the tensor element B_iB_j with the help of Eq. (13). Then the second moment is averaged over the ensemble. The averaging is performed by two steps: for the time intervals $(0,t)$ and $(t,t+\Delta t)$. This is possible if the statistical characteristics of the magnetic field vary slowly in comparison with the correlation time. The equation that describes the evolution of the second moment $W_{ij}(r,t) = \langle B_i(\mathbf{x},t)B_j(\mathbf{y},t)\rangle$ is obtained by means of this technique of Wiener integration and expansion in the short correlation time. It takes the form

$$\frac{\partial W_{ij}(t,\mathbf{r})}{\partial t} = T_{ik}^n\frac{\partial}{\partial r_k}W_{nj} + T_{jk}^n\frac{\partial}{\partial r_k}W_{in} + W_{nm}\frac{\partial}{\partial r_n}T_{ij}^m + \eta_{nm}\frac{\partial^2}{\partial r_n\partial r_m}W_{ij} , \qquad (14)$$

where

$$T_{ik}^n = V_{ik}^n(\mathbf{r}) - V_{ik}^n(0), \quad V_{ik}^n(\mathbf{r}) = \left\langle v_i(\mathbf{x})\frac{\partial v_k(\mathbf{y})}{\partial y_n}\right\rangle ,$$

$$\eta_{ij} = R_m^{-1}\delta_{ij} - f_{ij}(\mathbf{r}) + f_{ij}(0), \quad \mathbf{r} = \mathbf{x} - \mathbf{y} .$$

We consider this equation in the case statistically homogeneous, isotropic, and reflectionally invariant velocity and magnetic fields. It means that the correlation tensors in non-dimensional form for these fields are given by

$$\langle a_i(\mathbf{x})a_j(\mathbf{y})\rangle = \frac{1}{3}\left[A(r)\delta_{ij} + \frac{r}{2}A'\left(\delta_{ij} - \frac{r_ir_j}{r^2}\right)\right] , \qquad (15)$$

where $A(r) = F(r)$ for $a_i = u_i$, $A(r) = W(r)$ for $a_i = B_i$, and A' is the spatial derivative of A. The function $F(r)$ depends on a correlation function for the turbulent velocity field: $f(r) = [r^3F(r)]'/3r^2$. Note that the choice of the correlation function for the magnetic field in the form of Eq. (15) does not exclude the intermittency of the magnetic fluctuations. This is because the second moment of the turbulent magnetic field does not contain information on the structure of the typical distribution of the magnetic field. The principal contribution to the correlation function comes from widely spaced concentrations of the magnetic field. For a description of the intermittency of the magnetic fluctuations one needs to consider higher order moments [4].

Now let us take the trace of the tensor W_{ij} in Eq. (14), use Eq. (15) and the identities which are presented in the Appendix, and then multiply the resulting equation by r^2 and integrate it between the limits 0 and r. This procedure leads to the differential equation for the function W:

$$\frac{\partial W}{\partial t} = \frac{1}{m(r,N)}\left[W'' + \left(\frac{4}{r} - \frac{m'}{m}\right)W' - \frac{3m'}{2rm}W\right] .$$

The tensor element W_{ii} and the correlation function W are related by $W_{ii} = W + rW'/3$. The resulting equation describing the evolution of the correlation function W of the magnetic field takes a simpler form when it is formulated in terms of an auxiliary function $\Psi = r^2W/3\sqrt{2m}$. Then the equation reads

$$\frac{\partial \Psi}{\partial t} = \frac{1}{m(r,N)}\Psi'' - U\left(r, N, N', N'', \frac{\partial N}{\partial t}\right)\Psi . \qquad (16)$$

The nonlinearity N is defined by

$$N = W'' + \frac{4}{r}W' . \qquad (17)$$

The variable mass m and potential U are given by

$$U = \frac{2}{mr^2} - \frac{(m')^2}{4m^3} + \frac{\phi'}{r} - m\frac{\partial N}{\partial t} , \qquad (18)$$

$$m^{-1} = m_0^{-1}(r) + 2(N - N_*) , \qquad (19)$$

where we have used the definitions

$$m_0^{-1}(r) = 2R_m^{-1} + \frac{2}{3}[1 - F(r)] ,$$

$$\phi = \frac{[r^3(F - N)]'}{3r^2} ,$$

$$N_* = N(r = 0) .$$

Equation (16) is written in dimensionless variables: coordinates and time are measured in the units l_0 and l_0/u_0, in which l_0 is the main scale of turbulent hydrodynamic pulsations; it is of the same order as the density variation scale of neutral gas and u_0 is the characteristic value of the turbulent velocity $\mathbf{u}$. The magnetic field B and correlation function W are measured in units of b and b^2; $b = 4\pi enl_0u_0/\sqrt{3}c$. The magnetic Reynolds number $R_m = u_0l_0/\eta \gg 1$.

The nonlinear term $N(r, \Psi)$ in Eq. (16) is due to the Hall effect. N corresponds to a correlation function of the electric current

$$N = -\left(\frac{\sqrt{3}}{enu_0}\right)^2 \langle j(\mathbf{x},t) \cdot j(\mathbf{y},t)\rangle , \qquad (20)$$

where j is the projection of the electric current $\mathbf{j}$ on the direction $\mathbf{r} = \mathbf{x} - \mathbf{y}$.

We have assumed a stationary homogeneous, isotropic velocity field $\mathbf{u}$ [see Eq. (1)] which has a scale of order unity in normalized units. The exponential form could be replaced just as well by any rational function whose small r form is of the type $f(r) \approx 1 - r^2$. This is done for simplicity and to make the calculations more transparent for the reader. The theory remains valid for the most general case of velocity distribution. If $N = 0$, Eq. (16) is linear. In the next section we shall consider the linear theory of the magnetic fluctuations.

IV. THE LINEAR THEORY

Now let us consider the case $N = 0$. Equation (1) describes the evolution of the correlation function of the magnetic field and resembles the Schrödinger equation except for a variable mass and the absence of the imaginary unit in the time-derivative term. In the limit of large magnetic Reynolds number the formulation is amenable to treatment by a modified WKB method (see, e.g., [26]). We seek a solution to the linear equation for Ψ of the form $\Psi = \exp(2\gamma t)\Phi(r)$. The linear equation is reduced to the eigenvalue problem

$$\frac{1}{m_0(r)}\Phi'' - [2\gamma + U_0(r)]\Phi = 0 , \qquad (21)$$

with the boundary conditions $\Phi(r = 0) = \Phi(r = \infty) = 0$. Here the potential U_0 is

$$U_0 = \frac{2}{m_0r^2} - \frac{(m_0')^2}{4m_0^3} + \frac{f'}{r} . \qquad (22)$$

The spatial distribution of the function U_0 is shown in Fig. 2. The solution of Eq. (21) has the discrete (for $U_0 < 0$) and continuous (for $U_0 > U_*$) spectra, where the value U_* is shown in Fig. 2. The range of the quasidiscrete spectrum is located in $0 < U_0 < U_*$. The discrete

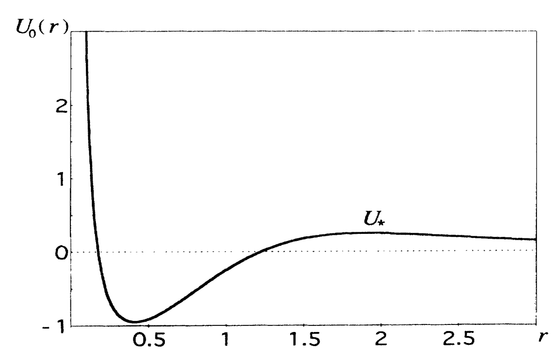

FIG. 2. The spatial distribution of the potential U_0.

spectrum describes the self-excitation of magnetic fluctuations, while the continuous spectrum corresponds to the dissipation of magnetic energy. The solution of Eq. (21) for the discrete spectrum in the vicinity of minimum of the potential well U_0 is considered early in [27,28]. In this paper we find solutions of Eq. (21) for the continuous spectrum and for the discrete spectrum near the threshold of the self-excitation of magnetic fluctuations (see also [30]).

The domain of definition of the function $\Phi(r)$ has three characteristic regions, in which the mass $m_0(r)$, the potential U_0, and Eq. (21) can be reduced to

$$\frac{1}{m_0} \sim \frac{2}{R_m}\,, \quad U_0 \sim \frac{4}{R_m r^2}\,, \quad \Phi'' - \frac{2}{r^2}\Phi = 0\,, \qquad 0 < r < aR_m^{-1/2} \tag{23}$$

$$\frac{1}{m_0} \sim \frac{2}{5}r^2\,, \quad U_0 \sim -\frac{8}{5}\,, \quad \Phi'' + \frac{\alpha}{r^2}\Phi = 0\,, \quad \alpha = 4 - 5\gamma\,, \qquad aR_m^{-1/2} < r < bR_m^{-1/4} \tag{24}$$

$$\frac{1}{m_0} \sim \frac{2}{3}\,, \quad U_0 \sim \frac{4}{3r^2}\,, \Phi'' - \left(\frac{2}{r^2} + 3\gamma\right)\Phi = 0\,, \qquad r \gg 1. \tag{25}$$

Here $a \sim b \sim 1$. It follows from Eqs. (23)–(25) that an asymptotic solution for the correlation function $W(r,t)$ is given by

$$W = W_* \exp(2\gamma t) \times \begin{cases} 1 & \text{for } 0 \le r < aR_m^{-1/2} \\ D_1 r^{-5/2}\cos(p\ln(r)/\ln(R_m) + \beta) & \text{for } aR_m^{-\frac{1}{2}} \le r < bR_m^{-\frac{1}{4}} \\ -D_2(\sqrt{3\gamma}r^{-2} + r^{-3})\exp(-\sqrt{3\gamma}r) & \text{for } r \gg 1, \end{cases} \tag{26}$$

where $p^2 = (\alpha - 1/4)\ln^2(R_m)$. The constants D_1, D_2, β and the growth (or damping) rate γ of the magnetic fluctuations can be determined after sewing the function W and the derivative W' at the boundaries of these regions. The general form of the growth (or damping) rate γ of the magnetic fluctuations is given by

$$\gamma = \frac{3}{4} - \frac{p^2}{\ln^2(R_m)}\,. \tag{27}$$

In the vicinity of the minimum of the potential well U_0 [for $p \ll \ln(R_m)$] the growth rate of the magnetic fluctuations is given by (27) with $p \simeq 2\pi k/\sqrt{5}$, $k = 1, 2, 3, \ldots$ (see [27,28]). The phase angle $\beta \simeq \pi/2 + \pi m, m = 0, \pm1, \pm2, \ldots$.

Near the threshold of the self-excitation of magnetic fluctuations ($\gamma \ll 1$) the parameter p in Eq. (27) is given by $p \simeq (2/\sqrt{5})[\pi k + g(\gamma)]$, $k = 1, 2, 3, \ldots$ (see [30]). Here $g(\gamma) \simeq 1$ is a slowly changing function. The phase angle $\beta \simeq \arctan[\ln(R_m)/2p] + \pi m$, $m = 0, \pm1, \pm2, \ldots$. A critical magnetic Reynolds number, which corresponds to $\gamma = 0$, is given by

$$R_m^{\rm cr} \sim \exp(2p/\sqrt{3})\,, \tag{28}$$

where $g(0) \approx 0.91$. For the main mode ($k = 1$) $R_m^{\rm cr} \approx 65.8$. This result agrees with the direct numerical simulation of the three-dimensional magnetohydrodynamics turbulence ($R_m^{\rm cr} \approx 65.0$; see [10]) and the numerical solution of Eq. (16) for $N = 0$ ($R_m^{\rm cr} \approx 66.0$; see [29]).

The damping rate of the magnetic fluctuations is given by

$$\gamma_d = -\frac{p^2}{\ln^2(R_m)}\,. \tag{29}$$

Here the parameter of the continuous spectrum $p \gg \ln(R_m)$. The phase angle $\beta \simeq \pi m$, $m = 0, \pm1, \pm2, \ldots$.

In the next section we shall obtain the nonlinear asymptotic and numerical solutions of Eq. (16) for the correlation function W of the turbulent magnetic field.

V. THE NONLINEAR ANALYSIS

Near the threshold of self-excitation of the magnetic fluctuations, the system tends to a steady state after several correlation times. It is also in agreement with the direct three-dimensional (3D) numerical simulation of the magnetic field generation by a prescribed Arnold-Beltrami-Childress (ABC) flow of the conducting fluids [31]. In particular, the Hall nonlinearity results in a saturation of the growth of the magnetic field. The magnetic Reynolds number threshold for the self-excitation of the magnetic field is $R_m \approx 66$ (see Sec. IV). First, it is important to find the stationary solution of Eq. (16).

The correlation function W satisfies the following boundary conditions: $W(r = 0) = W_*$ and $W(r = \infty) = 0$. Let us study the properties of the solution of Eq. (16) for small r, $r \ll 1$. The function Ψ and the nonlinearity N in this range can be reduced to

$$\Psi \approx \Psi_d r^2 + \cdots, \tag{30}$$

$$N \approx N_* + (N_d - 1/5)r^2 + \cdots\,. \tag{31}$$

The boundary condition $W(r = 0) = W_*$ and the definition $\Psi = r^2 W/3\sqrt{2m}$ yield $\Psi_d = W_*/(3\sqrt{R_m})$. The analysis of Eqs. (16) and (17) in the vicinity of $r = 0$ provides a relationship between N_* and W_*:

$$N_* = -10 R_m W_* N_d\,.$$

The function $\Psi(r)$ has three characteristic regions in which the mass, the potential, and Eqs. (16) and (17) can be reduced to

$$\frac{1}{m} \sim \frac{2}{R_m} + 2N_d r^2, \quad U \sim \frac{2}{mr^2}, \quad \Psi'' - \frac{2}{r^2}\Psi = 0 , \qquad 0 < r < aR_m^{-1/2} \quad (32)$$

$$\frac{1}{m} \sim \frac{2}{R_m} + 2N_d r^2, \quad U \sim -\frac{4(N_d+1)}{3}, \quad \Psi'' + \frac{2(N_d+1)}{3r^2}\Psi = 0 , \qquad aR_m^{-1/2} < r < r_0 \quad (33)$$

$$\frac{1}{m} \sim -2N_*, \quad U \sim -\frac{4N_*}{r^2}, \quad \Psi'' - \frac{2}{r^2}\Psi = 0 , r \gg 1. \quad (34)$$

Here we take into account that $|N_*| \gg 1$ and the point $r_0 = aR_m^{-\frac{1}{2}} \exp(\pi/2)$ corresponds to $W(r = r_0) = 0$. The asymptotic solution for the correlation function W of the magnetic fluctuations is given by

$$W = W_* \times \begin{cases} 1 & \text{for } 0 \le r < aR_m^{-1/2} \\ (a/r)^{\frac{5}{2}} R_m^{-\frac{5}{4}} \cos\{\ln[\sqrt{R_m}(r/a)]\} & \text{for } aR_m^{-\frac{1}{2}} \le r < r_0 \\ -Dr^{-3} & \text{for } r \gg 1. \end{cases} \quad (35)$$

Here $a \sim 1$ and $D \ll 1$. In the region $0 \le r < aR_m^{-\frac{1}{2}}$ the solution $W = W_* =$ const coincides with that of the linear problem [see Eqs. (23) and (26)]. It means that the strong nonlinearity (the large electric current $|N_*| \gg 1$) cannot suppress the generation of the magnetic fluctuations in this region. The magnetic field seems to be force free ($\mathbf{j} \times \mathbf{B} = 0$) in this region and so the flux tubes are twisted. In the second region ($aR_m^{-\frac{1}{2}} < r < r_0$) the function W decreases drastically; here the nonlinearity restricts the level of the magnetic fluctuations. The first and the second regions correspond to the flux rope. For $r > r_0$ the function W is negative.

Let us now discuss the significance of the negative correlation function. Consider a flux rope that is surrounded by other ropes. We examine the variation of the magnetic field as a function of distance from the flux rope. The magnetic fields from the other flux ropes cannot contribute to the correlation function W because $\langle \mathbf{B} \rangle = 0$. The only nonzero contribution to the function W is from the magnetic field of the flux rope itself. Outside the flux rope ($r > r_0$) the directions of the magnetic fields are opposite that inside the tube because of the condition $\nabla \cdot \mathbf{B} = 0$. The contribution from these magnetic fields for $r > r_0$ to the correlation function W is therefore negative.

The general properties of the nonlinear asymptotic solution (35) are confirmed by the numerical stationary solution of Eqs. (16) and (17). These equations can be reduced to

$$\frac{2}{m}\Psi'' = -\varphi + (U_0 + \tilde{U})\Psi , \quad (36)$$

$$2\Psi N'' = \varphi + (U_0 + \tilde{U})\Psi . \quad (37)$$

Here the functions φ and $\tilde{U}$ are given by

$$\tilde{U} = N\left(\frac{4}{r^2} + \frac{2}{9}mm_0(F')^2\right) - N'\left[\frac{4}{r} + \left(N' - \frac{2}{3}F'\right)\right],$$

$$\varphi = a_0\Psi + \frac{2}{3}F'\Psi' - \frac{r^2}{3\sqrt{2}m^{\frac{3}{2}}}N - b_0 N' ,$$

where

$$a_0 = \frac{m}{3}(F')^2 - \frac{2}{mr^2} + \frac{F''}{3}, \quad b_0 = 2\Psi' + m\Psi(2F'' - 3N').$$

At the point $r = r_0$ the function $\Psi = 0$. Therefore, Eq. (37) has a singularity at this point. In numerical simulation this point is passed analytically. The numerical stationary solutions for the correlation function W and the nonlinearity N for $R_m = 100$ are presented in Figs. 3 and 4. The solutions satisfied, the boundary conditions

$$W(r \to \infty) = 0 , \quad (38)$$

$$N(r \to \infty) = 0 \quad (39)$$

exist for $W_* = 0.36$ and $N_d = 0.99$. The obtained value of W_* is also in agreement with the direct 3D numerical simulation of the magnetic field generation by a prescribed ABC flow of the conducting fluids [31]. In particular, the saturation of the growth of the magnetic field by the

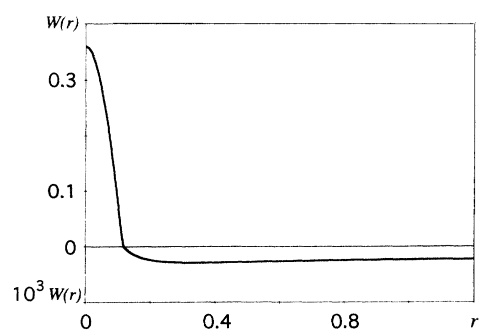

FIG. 3. Numerical stationary solution for the correlation function W of the magnetic field for $R_m = 100$.

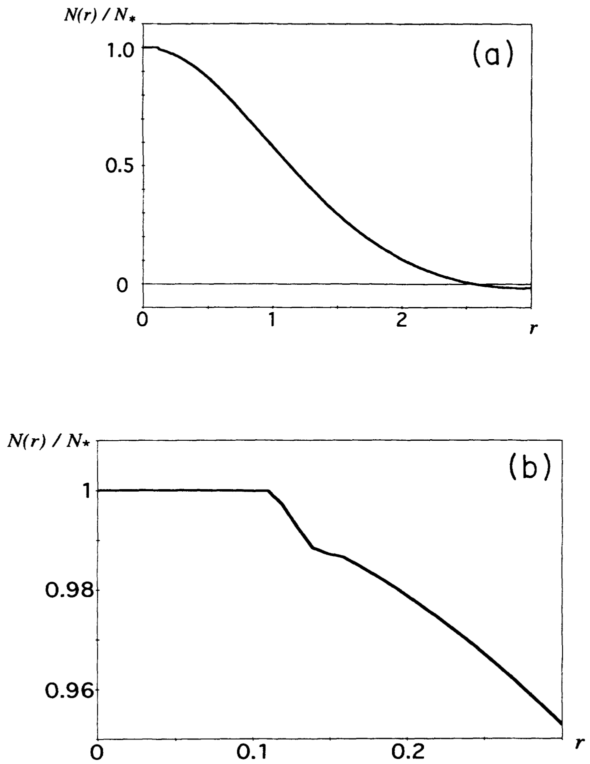

FIG. 4. Numerical stationary solution for the nonlinearity for $R_m = 100$. The function $N(r)/N_*$ is plotted at the vicinity $r = r_0$ with higher resolution (b).

Hall nonlinearity occurs at $W_* \approx 1$. A slight difference in W_* is due to the difference of the considered flows of the conducting fluids.

The correlation function $W = 0$ at the point $r = r_0 \approx 1.18 R_m^{-1/2}$. For $r > r_0$ the function W is negative. The correlation function W in the region $r > r_0$ is plotted in Fig. 3 scaled up by the factor $10^3 W$. This is done because the magnitude of the function W is very small in this region.

Note that the nonlinear solution is much more strongly localized than the linear one. This is a result of the strong dependence of the mass m on the distance r in the nonlinear case. In the vicinity of $r = r_0$ the mass drastically decreases from $R_m \gg 1$ to the value $R_m/|N_*|$. For $r \gg 1$ the mass is of the order of $|N_*|^{-1} \ll 1$. The dependence of the mass m on the distance r for the linear (curve 1) and nonlinear (curve 2) cases is shown in Fig. 5. The sharp drop of the mass in the nonlinear case determines the strong localization of the function W.

The question of intermittency cannot be treated in our framework because we have truncated the hierarchy of moment equations at the energy equation or second moment level. We point out, however, that the strong localization of the correlation function W indicates qualitatively a tendency toward intermittency, but we cannot quantify this further. Future studies will be oriented to extend the moment hierarchy and thus deal with the problems of energy flow and intermittency.

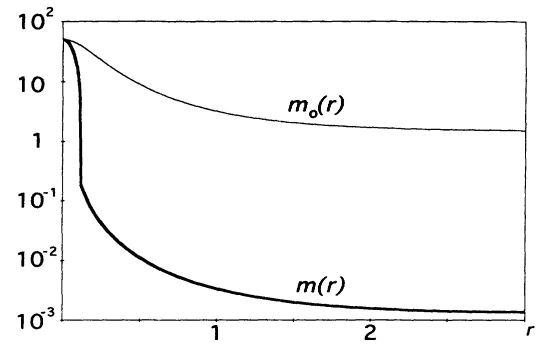

FIG. 5. The dependence of the mass on the distance r for the linear and nonlinear cases.

VI. DISCUSSION

The results obtained are of interest for the ionosphere of Venus. Filamentary magnetic structures were observed in the ionosphere of Venus [22]. The general properties of these small-scale magnetic field structures are as follows (see, e.g., [32,33]).

(i) Flux ropes are observed only in the regions where the average large-scale magnetic field is practically zero.

(ii) The flux ropes' axes have, in general, random inclinations.

(iii) The magnetic structures are stationary on the time scale on which the Pioneer Venus Orbiter spacecraft passes through them.

(iv) The cross section of the flux ropes is of the order of a few tens of kilometers.

Several mechanisms have been proposed to explain the origin of magnetic flux ropes [34–36]. The most difficult feature to understand, namely, the formation of the flux ropes in the absence of a large-scale regular magnetic field, has, however, remained unelucidated. The problem of their origin and evolution still remains, therefore, a subject of investigation, although the phenomenon was discovered in the ionosphere of Venus in 1979 [22].

According to the nonlinear theory considered in this paper the flux ropes observed in the ionosphere of Venus can be interpreted as magnetic fluctuations excited by random hydrodynamic flows of ionospheric plasma with zero mean magnetic field. On the basis of this theory we describe the general properties of the flux ropes in the ionosphere of Venus (see also [23]). In particular, the theory explains why flux ropes are not observed if there is a strong regular large-scale magnetic field, i.e., when the ionopause is low. The appearance of a strong regular interplanetary magnetic field in the ionosphere of Venus causes the turbulent flow of ionospheric plasma to become one dimensional and thus flux ropes cannot be generated. The characteristic lifetime of the turbulent flux ropes is of the order of the turnover time of turbulent eddies. In the ionosphere of Venus this time is of the order of several minutes. Despite of the random character of magnetic fluctuations, therefore, the spacecraft observations show a stationary picture of flux ropes (see also [32]). For the ionosphere of Venus $l_0 \approx (1-10) \times 10^6$ cm, $u_0 \approx (2-20) \times 10^3$ cm s^{-1}, $n \approx 10^4 - 10^5$ cm^{-3}, and $R_m \approx 70 - 150$. The characteristic cross section of the flux ropes is about $l_0 R_m^{-1/2} \geq 1 - 10$ km. It is in agreement with the observation of the flux ropes in the ionosphere of Venus [22].

The theory gives the mean square of the magnetic field $\langle B^2 \rangle$. The turbulent magnetic fields exist in the form of magnetic ropes separated by the regions with weak magnetic field. Let us estimate the maximum value of magnetic field inside the flux tube. The mean square of magnetic field is

$$\langle B^2 \rangle \approx V_s^{-1} \int B_m^2 \, dV_f \;, \tag{40}$$

where V_f is the volume of the flux rope, V_s is the volume of a turbulent eddy, and B_m is the magnitude of the magnetic field inside the flux tube. The volume are given by

$$V_f \approx l_0 (l_0 R_m^{-1/2})^2 \;, \quad V_s = l_0^3 \;. \tag{41}$$

Thus the maximum value of the magnetic field inside the flux tube is

$$B_m \approx \frac{4\pi e n \eta}{c} R_m^{3/2} \sqrt{W_*} \approx 50 - 100 nT \;, \tag{42}$$

which is consistent with observed values.

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APPENDIX

To derive Eq. (16) we use the following identities:

$$\delta_{nm} \frac{\partial^2}{\partial r_n \partial r_m} W_{ij} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial W_{ij}}{\partial r} \right),$$

$$\langle v_n v_m \rangle \frac{\partial^2}{\partial r_n \partial r_m} W_{ij} = \frac{\Phi(r)}{3r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial W_{ij}}{\partial r} \right),$$

$$\frac{\partial^2}{\partial r_n \partial r_k} W_{ii} = \frac{1}{r} \left[\delta_{kn} \frac{\partial W_{ii}}{\partial r} + r_n r_k \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial W_{ii}}{\partial r} \right) \right],$$

$$\frac{\partial f_{ik}}{\partial r_m} = \frac{1}{6r} \left[(4\delta_{ik} r_n - \delta_{in} r_k - \delta_{nk} r_i) \Phi' + r_n r^2 \left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \left(\frac{1}{r} \Phi' \right)' \right].$$

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